A Reduction Algorithm for Packing Problems

Michaël Gabay, Hadrien Cambazard, Yohann Benchetrit

A Reduction Algorithm for Packing Problems

Michaël Gabay[*], Hadrien Cambazard[*], Yohann Benchetrit[*]

July 15, 2014

Abstract: We present a reduction algorithm for packing problems. This reduction is very generic and can be applied to almost any packing problem such as bin packing, multi-dimensional bin packing, vector bin packing (with or without heterogeneous bins), etc. It is based on a dominance applied in the compatibility graph of a partial solution and can be computed in polynomial time in the input size and the number of bins, even on instances with high-multiplicity encoding of the input.

Keywords: Packing, Reduction algorithm, Multi-dimensional Packing, Vector Packing, Bin Packing, High-Multiplicity

1 Introduction

We are interested in combinatorial packing problems in general. There are usually two ways to solve such problems. The first one is to focus on the assignment of the items: one has to decide in which bin each item will be packed. The second one is focused on patterns: given all the feasible packings of items in bins, which patterns are used in an optimum solution and how many times? This second approach is the generalization of [1] approach for the cutting-stock problem. Pattern based exact algorithms are usually more efficient when the number of patterns is limited or there are items with large multiplicities while assignment based algorithms are usually better for heuristics and when the number of different items is large.

In this paper, we focus on assignment based approaches and propose a reduction algorithm based on a dominance property. In an assignment based solver for a packing problem, this property can be used to fix the assignment of many items at once and reduce the problem to a smaller packing problem. This is especially well suited to be integrated in branch-and-bound approaches for packing problems.

In packing problems, Dual Feasible Functions have been extensively used to obtain lower bounds [2, 3]. In this paper, we are interested in reduction procedures which ensure that optimality is preserved in the reduced problem. Such reductions have already been proposed by authors on specific packing problems, see e.g. [4–6]. [6] introduced the notion of Identically Feasible Function (IFF) to properly state the idea of reductions. The authors then propose IFF for 2-dimensional bin packing problems to remove small items and increase the size of large items. In [7], they investigate the use of tree-decomposition techniques to identify subproblems to be solved independently in a 2-dimensional packing problem with conflicts. Notice that reductions procedures or problem separation techniques were proposed very early by [4].

Reduction algorithms are often critical for the success of packing algorithms, see e.g. [8, 9] for presentation of reduction algorithms on the knapsack and the bin packing problems. In multi-dimension or with additional constraints, it is even more critical to be able to reduce packing problems to the smallest possible core problems. In this paper, we present a reduction algorithm

[*]Laboratoire G-SCOP 46, avenue Félix Viallet - 38031 Grenoble Cedex 1 - France

which can be used on whole instances of packing problems and can also be applied in the course of an assignment based algorithm for packing problems.

2 Definitions

We define a *pure packing problem* as a packing problem in which the capacities of the bins and the weights of the items are non-negative and the only constraints are the capacity constraints. The results presented in this paper are however much more general than this case. A number of other constraints can be added to the problem and if we simply take them into account when we compute the compatibility graph, then all the results for pure packing problems will still hold. For instance, we can add the following types of constraints:

- Variable item weights: if the weights of an item depends on the bin it is assigned to.
- Conflict constraints between items: if there are sets of conflicting items such that two items in a same set cannot be in a same bin.
- Incompatibility constraints between items and bins: if some items are incompatible with some bins, for instance because the items are fragile or heavy.

Basically, the results can be generalized to most constraints which do not involve both set of bins and set of items (for instance, spread or dependency constraints involve both set of bins and set of items).

A partial solution of a packing problem is a solution in which some but not all the items have been packed. Given a partial solution of a packing problem, observe that packing the remaining items is an instance of the same packing problem but in which bins have variable sizes. Pure packing problems have the nice property that for any feasible assignment, any subset of the bins and of the items in these bins is a partial solution of this problem and a feasible solution to the packing problem defined by only these bins and items. So if a partial solution is not feasible, then there is no feasible solution of the whole instance having this partial solution as a subset.

This is not necessarily true for "non-pure" packing problems such as the Machine Reassignment Problem[1]. In this problem, we also have to spread the items: a subset of a feasible solution may violate the spread constraint. However, the partial solution property holds for the underlying vector bin packing problem with heterogeneous bins.

We say that a partial solution P is feasible if it is a feasible solution to the pure packing problem defined with only the bins and items appearing in P. We say that it is *g-feasible* (g stands for globally) if P is a subset of a feasible solution to the underlying pure packing problem. Given a bin b (resp. a subset of bins X) we denote by b_P (resp. X_P) the set of items contained within this bin (resp. this subset of bins) in partial solution P.

Given a feasible partial solution, in the remaining subproblem, not all items fit into all bins. Let b be a bin, we say that an item $i \notin b_P$ is *compatible with* or *fits into* the bin b if the set of items $b_P \cup \{i\}$ fits into the bin b (starting from P, packing i into b gives another feasible partial solution).

Let $\mathcal{I}$ be the set of items and $\mathcal{B}$ be the set of bins. In a partial solution P, we denote by $\mathcal{I}_P$ the set of unpacked items and by $\mathcal{B}_P$ the set of bins in which at least one item from $\mathcal{I}_P$ fits. Let $X \subseteq \mathcal{B}_P$ be a subset of bins, we denote by $\Gamma(X) = \{i \in \mathcal{I}_P : \exists b \in X \text{ s.t. } i \text{ fits into } b\}$, the set of items which are compatible with these bins.

We have the following property:

Property 1. *Given a partial solution (possibly with no item assigned) P, for any $X \subseteq \mathcal{B}_P$ such that there is a feasible packing of $\Gamma(X)$ into X, let P_X be a partial feasible solution extending P and in which all items from $\Gamma(X)$ are assigned to X. P is g-feasible if and only if P_X is g-feasible.*

[1] http://challenge.roadef.org/2012/en/

Proof. If P_X is g-feasible, since P is a subset of P_X, P is obviously g-feasible.
Suppose P is g-feasible and consider a feasible solution S (S is a solution to the complete pure packing problem) which is an extension of P. Notice that bins from X contain only the items which they were assigned in P and some items from $\Gamma(X)$. If we remove the items in $\Gamma(X)$ from S, we obtain a partial solution P' which is clearly g-feasible. Moreover, in P', the bins from X are assigned exactly the same items as in P. So packing $\Gamma(X)$ into X is feasible in P'. We pack these items as in P_X and obtain a feasible solution S' containing P_X. Therefore, P_X is g-feasible. □

Property 1 gives a decomposition of packing problems into subproblems. It states that if we can find a subset of bins such that there is a feasible assignment of all of their compatible items within these bins, then we can pack these items in the bins and remove both the bins and items from further considerations. A valid set X is illustrated Figure 1, page 5.

Hence, if we can find such a set of bins and a feasible packing, we can assign all of them at once and reduce the problem to a smaller subproblem. However, given a set, determining whether it verifies the property is a hard matter. For instance, for $X = \mathcal{B}$ and $P = \emptyset$ we have the initial packing problem and even the (single dimension) bin packing problem is strongly $\mathcal{NP}$-hard and does not have a PTAS (unless $\mathcal{P} = \mathcal{NP}$ of course).

We propose a reduction algorithm based on flow-computation in the graph of items and bins compatibilities. Given a partial or an empty solution, it finds a feasible set X and a feasible assignment of the items from $\Gamma(X)$ into X. In Section 5, we present the reduction algorithm with single item assignments. It also proves that any packing problem in which the number of bins is greater than or equal to the number of items can be reduced in polynomial time to a packing problem with fewer bins than items. In Section 6, by using network flows, we generalize the results of Section 5 to more complex assignments. We present preliminary results and additional definitions in Section 4. In the next section, we expose a quick discussion explaining complexity matters in these problems and why the results presented in this paper are holding for both decision and optimization packing problems.

3 Discussion

We have not stated yet whether we are considering decision or optimization problems and which are the complexity parameters. In this section, we seek to clarify these matters.

3.1 Decision or Optimization ?

The problem description may let the reader think that we are focusing on decision problems. In decision problems, the number of bins is given. So checking bins' and items' compatibilities is straightforward. Moreover, we do not have to consider adding or removing bins: $\mathcal{B}$ is known in advance. In our case, we are considering both decision and optimization problems. In optimization problems, in any assignment based algorithm, if an item does not fit in any bin, then we have to add a new bin (the partial solution is not g-feasible with the current number of bins). So we can modify the algorithm to immediately add a new bin and extend $\mathcal{B}$ with it once an item is compatible with none of the bins. The results of the dominance properties and the reductions applied are not modified since adding a bin neither adds nor removes any edge from the other bins. One can argue that there are maybe several types of bins but, in this case, in exact algorithms we "only" have to branch to select which kind of bin is added. The algorithm can be adapted by simply branching earlier on the new bins. Moreover, in order to minimize the number of bins, many algorithms repeatedly solve the problem with a fixed number of bin. In this case, we are actually solving several decision problems.

In the following, we assume that the number of bins is fixed and equal to $|\mathcal{B}|$ but the results can easily be generalized to optimization problems.

3.2 Complexity

Regarding complexity, the algorithm is polynomial in the input size and in $|\mathcal{B}|$. If we have a high-multiplicity encoding of the input, it is legitimate to expect a high-multiplicity encoding of the output. For instance an output can be specified by giving the patterns used (each pattern being a way to fill a bin) and for each pattern how many items of each type are contained in the bin and how many times the pattern is used. Such an encoding is compact but not necessarily polynomial in the input size. Indeed, $|\mathcal{B}|$ is not necessarily polynomial in the input size when a high-multiplicity encoding is used. However, we are interested in algorithms here (and in solving $\mathcal{NP}$-hard problems…) and, above all, we are considering *assignment based algorithms*. Meaning that $|\mathcal{B}|$ is small enough to consider such approaches. Finally, observe that even assignment based algorithms can assign many items at once.

In the end, the algorithm may not be polynomial in the input size but it is polynomial in the size of the space which the programmer decided to allocate to his algorithm.

In the following, we assume that given a partial solution P, an item i and a bin b, we have access to an oracle which tells us whether i fits into b. However, we remark that for hyper-boxes geometric packing problems (strip packing, multi-dimensional bin packing,...) this problem is $\mathcal{NP}$-hard. Yet, for algorithms which do not consider reorganizing the contents of the bins in partial solutions, we can usually find out in polynomial time whether an item fits into a given bin.

4 Preliminaries

In the following, we use the bipartite compatibility graph of partial assignments. In this graph, each unpacked item is a vertex in the first partition and each bin in which at least one item fits is a vertex in the second partition. We denote this graph by $G_P = (\mathcal{I}_P, \mathcal{B}_P, E_P)$; the set of vertices is denoted by $V_P = \mathcal{I}_P \cup \mathcal{B}_P$. Let $u \in \mathcal{I}_P$ and $v \in \mathcal{B}_P$, the edge $\{u,v\}$ is in E_P if the item u fits into the bin v. Figure 1 illustrates the compatibility graph of a vector packing problem with two dimensions and also gives a set X satisfying Property 1. We remark that if we use constraint programming, the compatibility graph is obtained directly from the variables domains after filtering.

In a graph $G = (V,E)$, we denote by $\delta(u) = \{\{u,v\} : v \in V \text{ and } \{u,v\} \in E\}$ the set of edges incident to $u \in V$; we denote by $d(u) = |\delta(u)|$ the degree of a vertex $u \in V$ and by $\Gamma(X) = \{v \in V \setminus X : \exists u \in X \text{ s.t. } \{u,v\} \in E\}$ the neighbors of a set of vertices $X \subseteq V$. For singletons, we use $\Gamma(v)$ instead of $\Gamma(\{v\})$. In the following we will also use directed graphs but d, δ and Γ *always* refer to the (underlying) undirected graph.

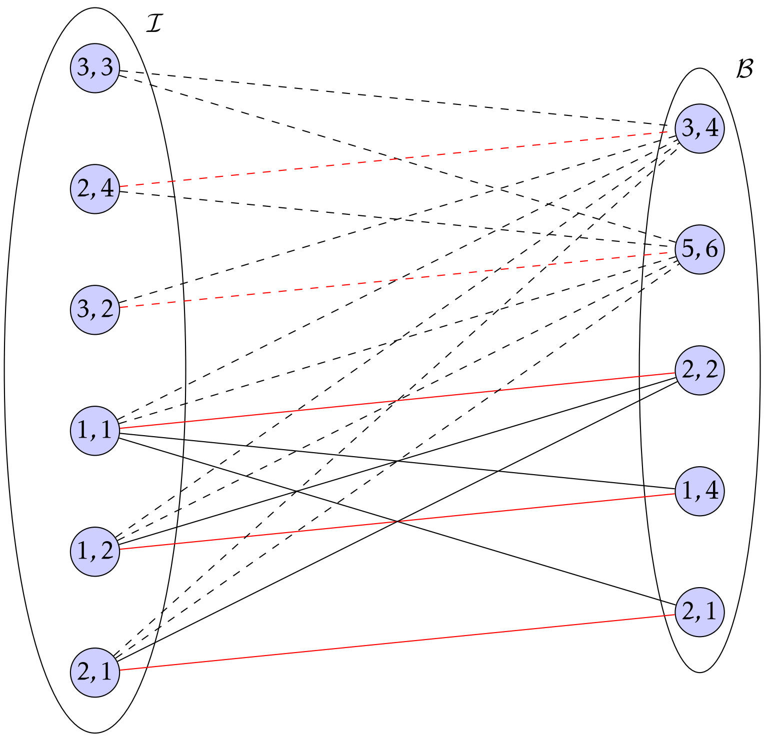
In the next section, we use Hall's theorem which is recalled below:

Theorem 1 (Hall's Theorem)**.** *A bipartite graph $G = (U,V,E)$ has a matching saturating U if and only if $\forall X \subseteq U$, $|X| \leq |\Gamma(X)|$.*

Now, let us consider the compatibility graph G_P for a given P and derive some basic properties from this graph. Let $u \in \mathcal{I}_P$, if $d(u) = 0$ then the item u cannot be packed. Hence, the partial solution P is g-infeasible. If $d(u) = 1$, then $\delta(u) = \{\{u,v\}\}$ for some $v \in \mathcal{B}_P$ and if the number of bins cannot be increased, the item u has to be packed into the bin v.

We said that we are only interested in bins in which at least one of the remaining item fits. Notice that if we added the other bins to the graph their degrees would be 0 so they would be isolated vertices which are modeling nothing useful. For a bin $v \in \mathcal{B}_P$, if $d(v) = 1$ then only one item $u \in \mathcal{I}_P$ can be assigned to the bin v. In any pure packing problem, it is dominant to pack this item into v. So we can assign item u to v and remove u and v from the compatibility graph.

Observe that, when an item is assigned to a bin, we can update the compatibility graph and ensure that all degrees are greater than or equal to 2 in linear time in the size of $\mathcal{I}_P$. When the graph is updated, we only remove edges and vertices but never add others. In the following,



The labels in the vertices from $\mathcal{I}$ are the requirements of the items in each dimension and the labels in the vertices from $\mathcal{B}$ are the remaining capacities of the bins in each dimension. Edges are denoting compatibilities. The three vertices $\{(2,2),(1,4),(2,1)\}$ from $\mathcal{B}$ form a set X satisfying Property 1. Plain edges are edges from $\delta(X)$, other edges are dashed. A maximum matching in this graph is given in red.

Figure 1: The bipartite compatibility graph of a vector packing problem.

we do not assume that all degrees are greater than 1 since it can result in loss of generality for optimization problems (when an item u is assigned because $d(u) = 1$).

We first present the reduction when there are no multiplicities on the items and at most one item is assigned to a bin. Then, we generalize the reduction algorithm to account for multiplicites on the items and perform reductions with assignments of more than one item into a bin. In this general case, the compatibility graph will be slightly different. We present the changes in Section 6.

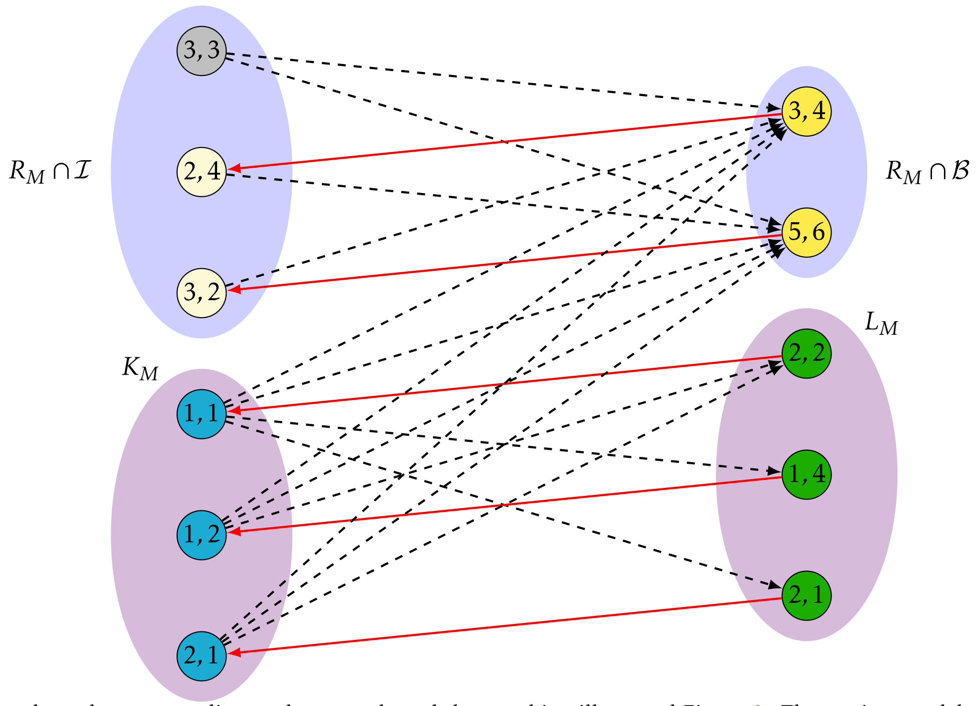
5 Matching-based reduction algorithm

In this section, we show that given the compatibility graph we can find a set X satisfying Property 1 and which is maximum for one-to-one assignments. A one-to-one assignment is an assignment in which each item is assigned to one bin and at most one item is assigned to a bin.

In order to find this set, we compute a maximum matching $M \subseteq E_P$ in the compatibility graph G_P. We orient the edges from G_P as follows: let $u \in \mathcal{I}_P$ and $v \in \mathcal{B}_P$ two adjacent vertices, i.e. $e = \{u, v\} \in E_P$, if $e \in M$ we orient e from v to u, otherwise e is oriented from u to v. We denote by $D_M = (\mathcal{I}_P, \mathcal{B}_P, A_M)$ the new directed graph. Let U_M be the set of all vertices from $\mathcal{I}_P$ which are not saturated by M, we denote by R_M the set of vertices reachable from U_M in D_M (we have $U_M \subseteq R_M$), and by $K_M = \mathcal{I}_P \setminus R_M$ the items which are unreachable from U_M and $L_M = \mathcal{B}_P \setminus R_M$ the bins which are unreachable from U_M. Figure 2 illustrates the orientation and these sets, in the oriented graph

obtained from the matching Figure 1.
We have the following result:

Theorem 2. *$L_M = \mathcal{B}_P \setminus R_M$ is a set satisfying Property 1 and the matching M gives a feasible assignment of $\Gamma(L_M)$ into L_M.*



The oriented graph corresponding to the example and the matching illustrated Figure 2. The vertices are labeled as in Figure 2. The gray vertex $(3,3)$ is the only node in the set U_M. All dashed edges are oriented from left to right and all red edges are edges from the matching and oriented from the right to the left.

Figure 2: An oriented compatibility graph

Proof. This proof is based on the proof from König's theorem [10]. König's theorem states that in a bipartite graph, the cardinality of a maximum matching is equal to the cardinality of a minimum vertex cover.

If M saturates $\mathcal{I}_P$, then $U_M = R_M = \emptyset$ and $L_M = \mathcal{B}_P$. Moreover, assigning each item to its corresponding bin in the matching is clearly a feasible assignment of all items.

Otherwise, M does not saturate $\mathcal{I}_P$. By definition of R_M, there is no arc from R_M to L_M. Moreover, there is no arc from L_M to R_M since any vertex from $R_M \cap \mathcal{I}_P$ is either not saturated by M (in U_M) or matched with a vertex in $R_M \cap \mathcal{B}_P$. So $\Gamma(L_M) \subseteq K_M$.

Finally, by definition of U_M and K_M, all vertices in K_M are saturated by M. Moreover, let $u \in \mathcal{I}_P$ and let $e = \{u, v\} \in M$ if $v \in R_M$, then by definition of R_M, $u \in R_M$. Hence, the matching M saturates all vertices of K_M using edges from $\mathcal{B}_P \setminus R_M = L_M$ to K_M. Therefore $K_M \subseteq \Gamma(L_M)$ and hence $\Gamma(L_M) = K_M$. Moreover, assigning each item from K_M to its corresponding bin in the matching is feasible since only one item is assigned to each bin and edges are denoting feasible assignments. □

Based on Theorem 2, we can compute a feasible set X of one-to-one assignments using Algorithm 1.

Compute a maximum matching M in G_P;
Compute R_M;
return $M, X = \mathcal{B}_P \setminus R_M$; `// The matching gives the assignment`

Algorithm 1: Matching-based reduction algorithm

The reduction assigns any item in K_M to its bin matched by M. Then, L_M and K_M are taken out of consideration. We denote by P' the new partial assignment. If K_M was empty, then $P' = P$. We have the following property:

Property 2. *The set X returned by Algorithm 1 is maximum for one-to-one assignments.*

Proof. Let $Q \in \mathcal{B}_P$ be a feasible set for Property 1 with one-to-one assignments. Observe that a one-to-one assignment is an assignment of one item into each bin, so it is a matching. Since Q is a feasible set, there is a one-to-one assignment, hence a matching, of $\Gamma(Q)$ into Q. Let W be a second feasible set for Property 1 with one-to-one assignments. Clearly, there is a matching saturating vertices from $\Gamma(W)$ with edges from $\Gamma(W)$ to W. Hence, there is a matching saturating vertices from $\Gamma(W \cup Q)$ with edges from $\Gamma(W \cup Q)$ to $W \cup Q$ and $|\Gamma(W \cup Q)| \leq |W \cup Q|$.

Therefore, in order to show that a feasible set X with one-to-one assignments is maximum, it is sufficient to show that it is maximal with respect to the union. Moreover, observe that for any set $W \in \mathcal{B}_P$, disjoint from Q, if $Q \cup W$ is a feasible set with one-to-one assignments, then there is a matching saturating vertices from $\Gamma(W) \setminus \Gamma(Q)$ with edges from W to $\Gamma(W) \setminus \Gamma(Q)$. Hence, $|W| \geq |\Gamma(W) \setminus \Gamma(Q)|$. Therefore, it is sufficient to show that once the reduction has been applied, in the new compatibility graph there is no set $X \subseteq \mathcal{B}_P$ such that $|X| \geq |\Gamma(X)|$.
Once the reduction has been performed and items from K_M have been assigned to the bins from L_M, we denote by P' the extended partial assignment and we have:

Lemma 1. *In $G_{P'}$, $\forall X \in \mathcal{B}_{P'}$, $|X| < |\Gamma(X)|$.*

Proof of Lemma 1. Let M' be the restriction of M to $G_{P'}$; M' is a maximum matching of $G_{P'}$. Clearly, M' is not a perfect matching and $L_{M'} = K_{M'} = \emptyset$.

Hence, the vertices of $\mathcal{B}_{P'} = R_{M'} \cap \mathcal{B}_{P'}$ are saturated by M'. By Hall's theorem, $\forall X \subseteq \mathcal{B}_{P'}$, $|X| \leq |\Gamma(X)|$. Suppose there exists $X \in \mathcal{B}_P$ such that $|X| \geq |\Gamma(X)|$, then $|X| = |\Gamma(X)|$ and X is saturated by M'. Therefore, M' is a perfect matching of $G_{{P'}_{|X \cup \Gamma(X)}}$ (the restriction of $G_{P'}$ to $X \cup \Gamma(X)$). Hence, $\Gamma(X) \cap U_{M'} = \emptyset$, there is no arc from $\mathcal{B}_{P'} \setminus X$ into $\Gamma(X)$ and obviously there is no arc from $\mathcal{I}_{P'} \setminus \Gamma(X)$ into X. Which contradicts $L_{M'} = \emptyset$ and $K_{M'} = \emptyset$. □

By Lemma 1, $\mathcal{B}_P \setminus R_M$ is maximal with respect to the union and hence, it is maximum for one-to-one assignments. □

As a consequence of Lemma 1, in a single run of the algorithm, we have proceeded to all feasible one-to-one assignments.

The algorithm computes a maximum matching and marks the vertices of the graph in a single pass. Therefore, the overall complexity is the complexity of the maximum matching algorithm which is $\mathcal{O}(|\mathcal{I}|^{2.5})$ if we use Hopcroft-Karp's matching algorithm [11].
A second consequence is the following:

Corollary 1. *For pure packing problems in which we can compute the compatibility graph in polynomial time: any instance with more bins than items ($|\mathcal{I}| \leq |\mathcal{B}|$) can be reduced to an instance with more items than bins ($|\mathcal{I}| > |\mathcal{B}|$) in polynomial time.*

On consecutive runs of the algorithm, the efficiency of the maximum matching algorithm can be improved by updating the previously computed maximum matching and using it for a hot start of the matching algorithm.

Observe that for the special case of the pure bin packing problem finding the set X is trivial. Indeed, an item which fits into a bin fits into all bins with smaller weight. Hence, the compatibility graph is (doubly) convex. The maximum matching can be obtained by simply sorting the bins and the items, or even in linear time with the algorithm from [12].

6 Generalized reduction algorithm

In this section, we extend previous results to other assignments than one-to-one assignments and instances which are specified using a high-multiplicity encoding of the input.

First, observe that a matching in a bipartite graph $G = (U, W, E)$ is an integer flow in the same graph extended with a source s connected to all vertices from U and a sink t connected to all vertices from W, and all edges from E being oriented from U to W. We now use this oriented compatibility graph $D = (V, A)$ (resp. $D_P = (V_P, A_P)$) in place of the previous one. We denote by $c(u, v)$ (resp. $c^P(u, v)$) the capacity of the arc (u, v) in the compatibility graph (resp. the compatibility graph of the partial solution P).

In order to generalize the results to high-multiplicity, we need to ensure that the size of the graph is polynomial in the input size and $|\mathcal{B}|$. In order to achieve this, we simply merge vertices from a same item type into a single vertex. Suppose that $\mathcal{I}$ is now the set of different item types and m_i the multiplicity of item $i \in \mathcal{I}$. In a partial solution P, we now consider that $\mathcal{I}_P$ is the set of remaining different item types and m_i^P denote the residual multiplicity of the item i (the number of items of type i which are not already packed in P). For an item $i \in \mathcal{I}$, we set $c(s, i) = m_i$ (resp. $c^P(s, i) = m_i^P$) and $c(i, b) = +\infty$ $\forall b \in \mathcal{B}$ with $(i, b) \in A$.

Now, suppose you expand this graph and replicate m_i times each vertex $i \in \mathcal{I}$. Let $b \in \mathcal{B}$ and $N = \{i \in \mathcal{I} : (i, b) \in A\}$ (since Γ is the neighborhood in the undirected graph, this is also $\Gamma(b) \cap \mathcal{I}$). We denote by κ_b the maximum number such that any subsets of items in N of size at most κ_b can be packed in b; $\kappa_b = \max\{k : \forall J \subseteq N \text{ with } |J| = k,\ J \text{ fits into } b\}$. We call κ_b the *robust capacity* of the bin b. The capacity of the arc (b, t) is set to $c(b, t) = \kappa_b$. We denote by $\kappa = (\kappa_1, \ldots, \kappa_{|\mathcal{B}|})$ the vector of robust capacities of the bins. We define κ_b^P and capacities of the arcs similarly for partial solutions. Similarly to one-to-one assignments, we define a κ-assignment as an assignment in which at most κ_b items are assigned to the bin b.

The compatibility graph now looks like the graph Figure 3. An example of a compatibility graph is illustrated Figure 4.

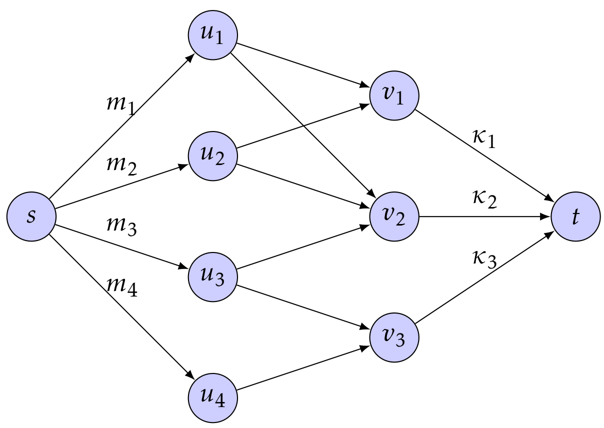


Figure 3: A generalized compatibility graph

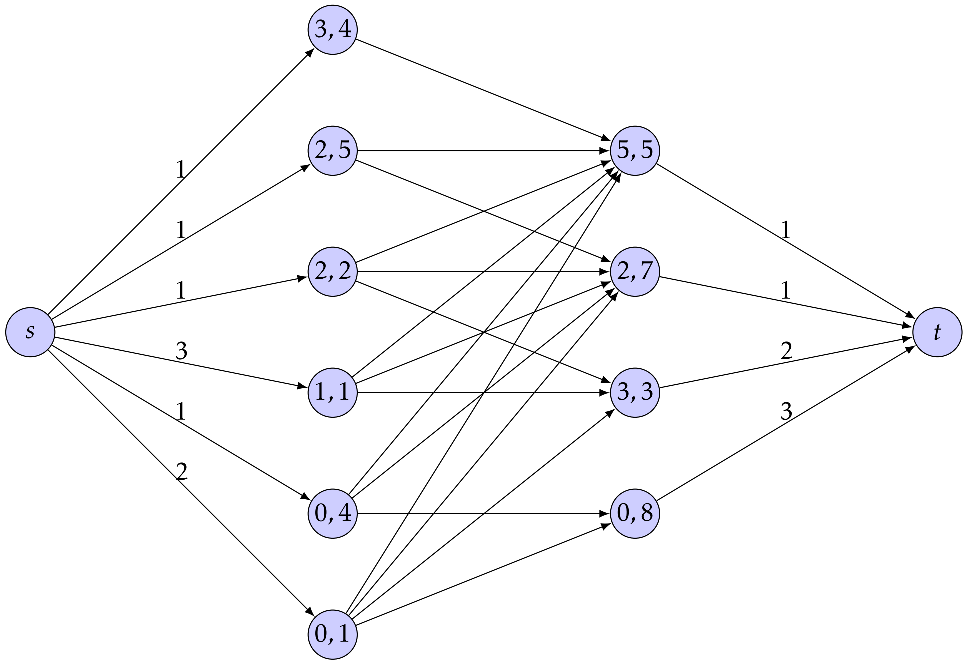


Figure 4: An example of a generalized compatibility graph for a 2-dimensional vector packing problem

Observe that determining the values of κ is easy for single dimension bin packing problems: Sort the items by decreasing order of the weights and let r be the remaining space in bin b. Let w_i be the weight of item i and $j = \min\{k : \sum_{i=1}^{k} m_i w_i > r\}$. Then, $\kappa_b = \sum_{i=1}^{j-1} m_i + \max\{k : k \leq m_j \text{ and } k \times w_j + \sum_{i=1}^{j-1} m_i w_i \leq r\}$.

For vector bin packing problems, we compute κ_b by applying the same procedure on all dimensions: $\kappa_b = \min \kappa_b^j$ where κ_b^j is the value of κ_b for the bin packing problem on dimension j.

For other packing problems, this problem can be $\mathcal{NP}$-hard but as we will see, only the diversity of combinations is depending on the values of κ's. So setting them to 1 will simply limit solutions to one-to-one assignments. Additionally, any heuristic giving a lower bound on the values of κ's will yield feasible reductions. So, for instance, in a 2-dimensional bin packing problem, if we create a rectangle whose width is the largest width among the items compatible with b and whose height is the largest height among the items compatible with b; then any heuristic giving a feasible number of such rectangles which can be packed, gives a lower bound on κ_b. And we can set the capacity $c(b, t)$ to this lower bound.

Finally, we can use fixed-parameter tractability results: when the number of items is fixed, for a set of items of the given size, we can usually determine in polynomial time whether the items fit. One can use such results when the number of compatible items is small but the number of combinations of a fixed number of items is quickly impracticable if we consider combinations of more than 3 items and bins with many compatible items.
Algorithm 2 gives the generalized reduction algorithm.

The construction used for Algorithm 1 is similar to the construction used in a proof from König's theorem [10]. König's theorem states that in a bipartite graph, the cardinality of a maximum matching is equal to the cardinality of a minimum vertex cover. Computing R_M as we did in Section 5 is the same as computing a minimum vertex cover from a maximum matching. In Algo-

Compute a maximum flow f in D_P;
Compute R_f, the set of all vertices reachable from s in the flow residual graph;
return $f, X = \mathcal{B}_P \setminus R_f$; `// The flow gives the assignment`

Algorithm 2: Generalized reduction algorithm

rithm 2, we compute the set of reachable vertices in the residual graph which is actually the same as computing a minimum cut based on a maximum flow. It is very natural to compute similar dual elements since König's theorem is a special case of the max-flow min-cut theorem.

In Algorithm 2, we mark R, the set of all vertices which can be reached from s in the residual graph. Clearly, R contains U, the set of unsaturated vertices from $\mathcal{I}$. Observe that with unit multiplicities, the set R is the same set as in Algorithm 1. We remark that with unit multiplicities, if we remove s, t and the edges with infinite capacities which have a positive flow from the residual graph, then the residual graph is exactly the directed graph defined in Section 5.

In fact, the proof of correctness of the algorithm is entirely based on Theorem 2.

Theorem 3. *Algorithm 2 gives a set X satisfying Property 1 and a feasible assignment. Moreover, X is maximum for κ-assignments.*

Proof. From D_P we create the following graph $G'_P = (I', B', E)$: let I' be the set of vertices in which each vertex $i \in \mathcal{I}_P \cap V$ is replicated $c(s,i)$ times; let B' be the set of vertices in which each vertex $b \in \mathcal{B}_P \cap V$ is replicated $c(b,t)$ times; s and t do not belong to G'_P. Let $E = \{\{u,v\} : (u,v) \in A \text{ or } (v,u) \in A\}$. A maximum flow f in D immediately gives a maximum matching M in G'_P by dividing the flow in units.

Clearly, if $\forall b \in \mathcal{B}_P\ \kappa_b = 1$, then $G'_P = G_P$, the compatibility graph defined by the same instance in which items are replicated instead of given with multiplicities. So Theorem 2 proves the theorem.

If κ is not a 1 vector, then the bins are also replicated. Since the bin b is replicated exactly κ_b times and any combination of κ_b items, which are compatible with b, fits into b, the assignment given by the flow is feasible and verifies Property 1. Moreover, any κ-assignment of k items can be divided in k one-to-one assignments of one item into a bin by replicating each bin at most a number of times equal to its robust capacity, and conversely. Moreover, the assignment obtained in G'_P is a maximum one-to-one assignment by Theorem 2 therefore it is a maximum κ-assignments in D_P. □

The complexity of Algorithm 2 is equal to the complexity of computing a maximum flow in D_P. This is polynomial in the size of D_P whose size is polynomial in the instance size and $\mathcal{B}$. In practical implementations, we can keep previously computed flows for a hot start of the maximum flow algorithm.

Finally, we can generalize Corollary 1:

Corollary 2. *For pure packing problems in which we can compute the compatibility graph in polynomial time: any instance s.t. $\sum_{b\in\mathcal{B}} \kappa_b \geq \sum_{i\in\mathcal{I}} m_i$ can be reduced to an instance with $\sum_{b\in\mathcal{B}'} \kappa_b < \sum_{i\in\mathcal{I}'} m_i$ and $\mathcal{B}' \subset \mathcal{B}, \mathcal{I}' \subseteq \mathcal{I}$, in polynomial time in the input size and the number of bins, even with high-multiplicity encoding of the input.*

References

[1] Paul C Gilmore and Ralph E Gomory. A linear programming approach to the cutting-stock problem. *Operations research*, 9(6):849–859, 1961.

[2] François Clautiaux, Cláudio Alves, and José Valério de Carvalho. A survey of dual-feasible and superadditive functions. *Annals of Operations Research*, 179(1):317–342, 2010.

[3] Cláudio Alves, José Valério de Carvalho, François Clautiaux, and Jürgen Rietz. Multidimensional dual-feasible functions and fast lower bounds for the vector packing problem. *European Journal of Operational Research*, 233(1):43–63, 2014.

[4] Silvano Martello and Paolo Toth. Lower bounds and reduction procedures for the bin packing problem. *Discrete Applied Mathematics*, 28(1):59–70, 1990.

[5] Peter A. Huegler and Joseph C. Hartman. Problem reduction for one-dimensional cutting and packing problems. Technical report.

[6] Jacques Carlier, François Clautiaux, and Aziz Moukrim. New reduction procedures and lower bounds for the two-dimensional bin packing problem with fixed orientation. *Computers & Operations Research*, 34(8):2223–2250, 2007.

[7] Ali Khanafer, François Clautiaux, and El-Ghazali Talbi. Tree-decomposition based heuristics for the two-dimensional bin packing problem with conflicts. *Computers & Operations Research*, 39(1):54–63, 2012.

[8] Silvano Martello and Paolo Toth. *Knapsack problems*. Wiley New York, 1990.

[9] Hans Kellerer, Ulrich Pferschy, and David Pisinger. *Knapsack problems*. Springer, 2004.

[10] D. König. Graphs and matrices (in hungarian). *Mat Fiz Lapok 38*, pages 116–119, 1931.

[11] John E Hopcroft and Richard M Karp. An nˆ5/2 algorithm for maximum matchings in bipartite graphs. *SIAM Journal on computing*, 2(4):225–231, 1973.

[12] G Steiner and JS Yeomans. A linear time algorithm for maximum matchings in convex, bipartite graphs. *Computers & Mathematics with Applications*, 31(12):91–96, 1996.